TECHNICAL NOTES

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS.

No. 139.

INFLUENCE OF RIBS ON STRENGTH OF SPARS.

By L. Ballenstedt.

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INFLUENCE OF RIBS ON STRENGTH OF SPARS.*

By L. Ballenstedt.

In calculating the strength of airplane wing spars, the assumption is usual that the ribs are connected to the spars by flexible joints. This assumption is not accurate, as the ribs are attached very firmly to the spars by means of brackets, nails and glue. This method of attachment is so rigid, with reasonably good workmanship, that it is justifiable to assume that the ribs are rigidly attached to the spars.

The aim of the following investigation is to determine what effect this type of joint has on the strength of the spars. The investigation was suggested by the striking fact that the practical loading tests generally gave greater strength and smaller deflection than strength calculations based on the assumption of ribs attached by flexible joints. The difference was particularly noticeable, with heavily offset loading and arises from the fact that the more heavily loaded spar transmits a portion of its load through the ribs to the more lightly loaded spar.

Fig. 1 shows the framework of the wing in perspective. It consists of two spars with nine ribs, and rests on points A - B - C - D. A statically determinate system is produced, when all the ribs are cut through with the exception of the rib A - C (hatched in Fig. 1). Two simple spars A - B and C - D are thus obtained.

* From Technische Berichte, Volume III, No. 4, pp. 100-107, (1918).

The Rib A – C is required in order to prevent the spars from turning about their longitudinal axis. For symmetry, the central rib might be considered as part of the statically determinate system and the rib A – C could then be cut, but this is not of advantage in the computation, since it makes the determination of the displacement δ_{aa}, δ_{ab} .... etc. more complicated. At the points of section of the ribs, three unknowns generally appear: longitudinal force, shearing force and bending moment, and the system is, therefore, $3 \times 8 = 24$-fold statically indeterminate.

In order not to complicate the investigation unnecessarily, the following assumptions may be made, viz:

1. The external forces act at right angles to the plane through the longitudinal axis of the spars. The longitudinal forces in the (straight) ribs will then be zero.

2. The moments of inertia and the areas of the two spar sections are equal and the external forces act only on the spars, not on the ribs. The bending moments in the central portions of the ribs thus become zero. The strict proof of this is put in the Appendix in order not to interrupt the course of the analysis. Besides, the arrangement shows at once that the elastic lines of the ribs must have a point of flexion at the center i, Fig. 11, since the angle of torsion $\Delta\theta$, of corresponding cross-sections of the two spars must be equal with equal cross-section and without load on the ribs.

The wing frame is now only eight-fold statically indeterminate.

Let the shearing forces acting at the center of the ribs be designated by X_a, X_b, X_c, etc. With rigid supports, these must satisfy the elastic conditions-*

$$1.\quad X_a\ \delta_{aa} + X_b\ \delta_{ab} + X_c\ \delta_{ac} + \ldots . X_h\ \delta_{ah} = \Sigma P_m\ \delta_{ma}$$

$$2.\quad X_a\ \delta_{ab} + X_b\ \delta_{bb} + X_c\ \delta_{bc} + \ldots . X_h\ \delta_{bh} = \Sigma P_m\ \delta_{mb}$$

$$3.\quad X_a\ \delta_{ac} + X_b\ \delta_{bc} + X_c\ \delta_{cc} + \ldots . X_h\ \delta_{ch} = \Sigma P_m\ \delta_{mc}$$

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$$8.\quad X_a\ \delta_{ah} + X_b\ \delta_{bh} + X_c\ \delta_{ch} + \ldots . X_h\ \delta_{hh} = \Sigma P_m\ \delta_{mh}$$

1. Calculation of coefficients of the unknowns.

With loads $X_a = -1$, $X_b = -1$, $X_c = -1$, etc., etc. ribs are subjected to bending, and the spars to bending and torsional stresses (Fig. 2).

In general, let -

M_p	denote	the	bending	moment	resulting	when	$X_p = -1$
M_q	"	"	"	"	"	"	$X_q = -1$
T_p	"	"	torsional	"	"	"	$X_p = -1$
T_q	"	"	"	"	"	"	$X_q = -1$

for any section of the wing framework.

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* Muller-Breslau: "Die neueren Methoden der Festigkeitslehre und der Statik der Baukonstruktionen" (Recent methods of the theory of the statical strength of framed structures), T.B. 1913, p. 209.

Further, let -

I denote the equatorial moment of inertia of the spars,

I_1 " " " " " " " " ribs,

I_p " " polar moment of inertia of the spars,

E " " modulus of elasticity,

S " " " " shearing,

and we get -

$$\delta_{pq} = \int \frac{M_p \quad M_q}{E \quad I} dx + \int \frac{T_p \quad T_q}{S \quad I_p} dx + \int \frac{M_p \quad M_q}{E \quad I_1} dz,^*$$

where the first two integrals are taken over both spars and the last one over all the ribs. The section of the spar has, for the sake of simplicity, been assumed to be circular or annular. If this is not the case, it becomes necessary to introduce the expression -

$$\zeta \frac{I_p}{4 \, I_x \, I_y} \text{ or, according to Saint Venant, } \frac{F^4}{40 \, I_p}$$

in place of I_p.*

On the assumption, which corresponds with actual conditions, that

$$I_1 = \frac{1}{6} I; \quad I_p = 2 I; \quad S = \frac{3}{8} E$$

we obtain -

$$E \, I \, \delta_{pq} = \int M_p \quad M_q \quad dx + \frac{4}{3} \int T_p \quad T_q \quad dx + 6 \int M_p \quad M_q \quad dz.$$

The evaluation of $E \times I \times \delta_r$ follows at once from Figs. 3 to

* Ibid p. 211 and 255.
** "Hutte" 22nd edition, Vol. I, pp. 570-1.

7, in conjunction with Table 1. The calculation of $E \times I \times \delta_{bd}$ will be exhibited here for easy comprehension. The bending and torsional moments produced in the spars under the conditions, $X_b = -1$, $X_d = -1$, are indicated in Figs. 4 and 6. Fig. 7 shows the bending moment in the rib b or d, and in the end rib A - C. The distance between the ribs is taken as s, and the distance between the spars at the same time as 2s. By reason of the load $X_b = -1$, pressures arise in the spars of the magnitudes $\pm \frac{6}{8}$ and $\pm \frac{2}{8}$ and the bending moments on the spars, for the parts from A and C as far as to the rib b, are, therefore:

$$M_b = \pm \frac{6}{8} x$$

from rib b to rib d:

$$M_b = \pm \left[\frac{6}{8} 2s + \left(\frac{6}{8} - 1 \right) \xi \right] = \pm \left[\frac{12}{8} s - \frac{2}{8} \xi \right]$$

from rib d to rib h:

$$M_b = \pm \frac{2}{8} x'$$

In the same manner, from $X_d = -1$ (Fig. 6) for the parts from A and C up to rib b, there arises the moment:

$$M_d = \pm \frac{4}{8} x$$

From rib b to rib d:

$$M_d = \pm \frac{4}{8} (2s + \xi)$$

From rib d to rib h:

$$M_d = \pm \frac{4}{8} x'$$

Thus we obtain:

$$\int M_p \quad M_q \quad dx = 2 \int_0^{l} M_b \quad M_d \quad dx =$$

$$= 2\left[\int_0^{2s} \frac{6}{8} x \quad \frac{4}{8} x \quad dx + \int_0^{2s} \left(\frac{12}{8} s - \frac{2}{8} \xi\right) \frac{4}{8} (2s + \xi) \, d\xi \, + \right.$$

$$\left. + \int_0^{4s} \frac{2}{8} x' \, \frac{4}{8} x' \quad dx' \right] = \frac{176}{12} s^3.$$

The torsional moments of the spars arising from $X_b = -1$ are:-

For the parts from A and C to rib b:

$$T_o = \pm s$$

From rib b to rib h:

$T_b = 0$ (See Fig. 4).

and from $X_d = -1$

for parts from A and C to rib d:

$$T_d = \pm s$$

From rib d to rib h

$$T_d = 0$$

Hence -

$$\frac{4}{3} \int T_p \quad T_q \quad dx = 2 \, \frac{4}{3} \int_0^{2s} s \quad s \quad dx = \frac{16}{3} s^3.$$

and finally from $X_b = -1$, there arises in the rib b and in the end rib AC the bending moment $M_b = \pm z$ (compare Fig. 7).

For the remaining ribs we find $M_b = 0$.

In the same manner, we obtain with $X_d = -1$ in the rib d and in the end rib AC: $M_d = \pm z$ and in the remaining ribs, $M_d = 0$.

We therefore, get -

$$6 \int M_p \; M_q \; dz = 2 \quad 6 \int_0^s z \quad z \quad dz = 4s^3$$

The integral extends here only over the end rib because $M_p \times M_q = 0$ for all the other ribs. Thus, we obtain -

$$E \quad I \quad \delta_{bd} = \frac{176}{12} s^3 + \frac{16}{3} s^3 + 4s^3 = \frac{288}{12} s^3$$

The other coefficients of the unknowns given in Table 1 have been determined in a similar manner.

2. Determination of $\Sigma P_m \times \delta_{mq}$.

In the case where the same load P acts at each node of a spar, we have in general: $\Sigma P_m \; \delta_{mq} = P \; \Sigma \delta_{mq}$. Since the loads P act at the same points of the spars as the shearing forces X_a, X_b, X_c, etc., and as they only produce bending moments in the one spar, the values δ_{mq} may be taken direct from Table 1, column 1.

$$E \quad I \quad \Sigma P_m \quad \delta_{ma} = \frac{P}{2 \times 12} (49 + 81 + 95 + 94 + 81 + 59 + 31) \; s^3 = \frac{245}{12} P \; s^3,$$

$$E \quad I \quad \Sigma P_m \quad \delta_{mb} = \frac{P}{2 \times 12} (81 + 144 + 175 + 176 + 153 + 112 + 59) \; s^3 = \frac{450}{12} P \; s^3,$$

$$E \quad I \quad \Sigma P_m \quad \delta_{mc} = \frac{P}{2 \times 12} (95 + 175 + 225 + 234 + 207 + 153 + 81) \; s^3 = \frac{585}{12} P \; s^3,$$

$$E\ I\ \Sigma P_m\ \delta_{md} = \frac{P}{2 \times 12}(94 + 176 + 234 + 256 + 234 + $$
$$+ 176 + 94)\ s^3 = \frac{632}{12} P\ s^3;$$

$$E\ I\ \Sigma P_m\ \delta_{me} = E\ I\ \Sigma P_m\ \delta_{mc} = \frac{585}{12} P\ s^3,$$

$$E\ I\ \Sigma P_m\ \delta_{mf} = E\ I\ \Sigma P_m\ \delta_{mb} = \frac{450}{12} P\ s^3,$$

$$E\ I\ \Sigma P_m\ \delta_{mg} = E\ I\ \Sigma P_m\ \delta_{ma} = \frac{245}{12} P\ s^3,$$

$$E\ I\ \Sigma P_m\ \delta_{mh} = 0.$$

3. Calculation of the unknowns.

We obtain:

1. $177X_a + 161X_b + 175X_c + 174X_d + 161X_e + 139X_f +$
   $+ 111X_g + 80X_h = 245P,$

2. $161X_a + 304X_b + 287X_c + 288X_d + 265X_e + 224X_f +$
   $+ 171X_g + 112X_h = 450P,$

3. $175X_a + 287X_b + 417X_c + 378X_d + 351X_e + 297X_f +$
   $+ 225X_g + 144X_h = 585P,$

4. $174X_a + 288X_b + 378X_c + 480X_d + 410X_e + 352X_f +$
   $+ 270X_g + 176X_h = 632P,$

5. $161X_a + 265X_b + 351X_c + 410X_d + 481X_e + 383X_f +$
   $+ 303X_g + 208X_h = 585P,$

etc.

$$8. \quad 80X_a + 112X_b + 144X_c + 176X_d + 208X_e + 240X_f + $$
$$+ 272X_g + 352X_h = 0.$$

Since, from symmetry, $X_g = X_a$, $X_f = X_b$, $X_e = X_c$, the first five equations are sufficient to determine the unknown. Equation 8 serves as a check.

$$1. \quad 288X_a + 300X_b + 336X_c + 174X_d + 80X_h = 245P,$$

$$2. \quad 332X_a + 528X_b + 552X_c + 288X_d + 112X_h = 450P,$$

$$3. \quad 400X_a + 584X_b + 768X_c + 378X_d + 144X_h = 585P.$$

$$4. \quad 444X_a + 640X_b + 788X_c + 480X_d + 176X_h = 632P,$$

$$5. \quad 464X_a + 648X_b + 832X_c + 410X_d + 208X_h = 585P,$$

$$8. \quad 2X_a + 2X_b + 2X_c + 2X_d + 2X_h = 0.$$

Since the shearing force in the uncut rib is equal to X_h, owing to the symmetry, it would have been possible to obtain equation 8 from the equations for moments in the longitudinal axis of a spar.

We have from the equations 1 to 5:

$$X_d = + 0.544 \text{ P},$$

$$X_c = + 0.497 \text{ P},$$

$$X_b = + 0.315 \text{ P},$$

$$X_a = - 0.117 \text{ P},$$

$$X_b = - 0.967 \text{ P}.$$

These values put in equation 8 give:

$$2\,(0.497 + 0.315 - 0.117 - 0.967) + 0.544 = 0,$$

$$2\,(-\,0.272) + 0.544 = 0,$$

$$0 = 0.$$

Then the forces $P - X$ act on the loaded spar, so that:

$$P_d = +\,0.456\ P,$$

$$P_c = +\,0.503\ P,$$

$$P_b = +\,0.685\ P,$$

$$P_a = +\,1.117\ P,$$

$$P_h = +\,1.967\ P.$$

4. Bending moments and stresses.

In Fig. 8, the forces on the loaded spar are plotted as ordinates and the end points joined by a smooth curve. The curve resembles a parabola. The horizontals which unite the end points of the loads P were drawn for comparison. It is known that the load is transferred from the center to the supports on account of the reaction of the ribs. The maximum bending moment is

$$M_{max} = P\ s\ (2.533 \times 4 - 1.117 \times 3 - 0.685 \times 2 - 0.503 \times 1) =$$
$$= 4.908\ P\ s$$

against -

$$M'_{max} = P\ s\ (3.5 \times 4 - 3 - 2 - 1) = 8.000\ P\ s.$$

The maximum bending moment, therefore, only amounts roughly to $\frac{49}{80}$ of that obtained when the effect of the ribs is neglected.

The areas of both moments have been plotted for comparison in Fig. 9. Fig. 10 shows the course of the moments of torsion on each of the two spars.

The main stress is only slightly increased by the torsional moments as will be seen from the following calculation.

For a circular section, for instance, in the section a-b where the torsional stress is greatest, there arises,

the normal stress: $\sigma = \frac{3.949}{I} P\ s\ r$

the shearing stress: $\tau = \frac{1.084}{2\ I} P\ s\ r$

so that $\tau = \frac{1.084}{2 \times 3.949} \sigma = 0.137\ \sigma.$

With $m = 3$ and $\alpha_0 = 1$* we obtain the principal stress:

$$\sigma'_{max} = 0.333\ \sigma + 0.667 \sqrt{\sigma^2 + 4 \times 0.137^2\ \sigma^2} = 1.025\ \sigma .$$

For the section at the center of the spar we get:

$$\tau = \frac{0.272}{2 \times 4.908} \sigma = 0.0277\ \sigma,$$

$$\sigma'_{max} = 0.333\ \sigma + 0.667 \sqrt{\sigma^2 + 4 \times 0.0277^2\ \sigma^2} = 1.001\ \sigma .$$

The increase of the principal stresses is, therefore, insignificant both in the section of maximum torsion (2.5%) and in the section of maximum bending moment (0.1%).

The bending moments in the ribs are also of such magnitude that they can well be obtained from the present sections. In the worst case, in the rib h -

$$M = 0.967\ P\ s = \sim \frac{1}{8} M'_{max}.$$

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* Compare "Hutte" 22nd edition, Volume 1, p. 527.

Since the moment of resistance of the ribs is generally 1/6 to 1/8 of the moment of resistance of the spars, there is no danger of the ribs being overloaded.

5. Conclusion.

The above investigation demonstrates that the loads are distributed between the spars in a satisfactory manner by means of ribs. In general, this results in an increase in strength, since in most cases the external forces act very unequally on the two spars. If the forces act only on one spar, or on one spar upwards and on the other spar downwards, then the gain is considerable, in the present instance about 40%. The investigation only proves this, however, for simple bending, but similar conclusions may be drawn for buckling, as the deflections will be diminished in the same proportion as the bending moments.

This favorable result brings up the question, whether in selecting methods for calculating airplanes, with the object of approximating the actual stresses as closely as possible, the results will not be too unfavorable. The calculation of stresses with a multiple of the load, for instance, is not used for other purposes, not even in bridge building, where safety is as important as in an airplane.

Would it not be sufficient if factors of safety were determined from the stresses of unitary load? The breaking tests with wings show that the stresses calculated with unitary load come

nearest to the actual conditions.

Apart from the fact that the calculations with a multiple of the load frequently lead to impossible dimensions of the spar, the diminished work, at least in the design of new airplanes, should not be underestimated. But even theoretically, it is more correct to use the rules and formulas within the limits of proportionality for which alone they hold good.

Appendix:- For the case where the moments at the center of the ribs are not zero, there must be introduced for each section a, b, c, ... h, a moment X'_a, X'_b, X'_c, ... X'_h. The elasticity equations, under the same assumptions as before, are then

1. $X_a \; \delta_{aa} + X'_a \; \delta_{a'a} + X_b \; \delta_{ba} + X'_b \; \delta_{b'a} + X_c \; \delta_{ca} +$

$$+ X'_c \; \delta_{c'a} + \ldots = \Sigma P_m \; \delta_{ma},$$

2. $X_a \; \delta_{aa'} + X'_a \; \delta_{a'a'} + X_b \; \delta_{ba'} + X'_b \; \delta_{b'a'} + X_c \; \delta_{ca'} +$

$$+ X'_c \; \delta_{c'a'} + \ldots = \Sigma P_m \; \delta_{ma'},$$

3. $X_a \; \delta_{ab} + X'_a \; \delta_{a'b} + X_b \; \delta_{bb} + X'_b \; \delta_{b'b} + X_c \; \delta_{cb} +$

$$+ X'_c \; \delta_{c'b} + \ldots = \Sigma P_m \; \delta_{mb}$$

4. $X_a \; \delta_{ab'} + X'_a \; \delta_{a'b'} + X_b \; \delta_{bb'} + X'_b \; \delta_{b'b'} + X_c \; \delta_{cb'} +$

$$+ X'_c \; \delta_{c'b'} + \ldots = \Sigma P_m \; \delta_{mb'}$$

. . . .

etc. to 16.

If i denotes the point of application of any shearing force X_i and k' denotes the point X'_k, we have, as above:

$$\delta_{ik'} = \int \frac{M_i \quad M_{k'}}{E \quad I} dx + \int \frac{T_i \quad T_{k'}}{S \quad I_p} dx + \int \frac{M_i \quad M_{k'}}{E \quad I_1} \delta z$$

The moment $X_{k'} = -1$ produces the moments $M_{k'}$ and $T_{k'}$ for both halves of the wing with the same sign. The shearing force $X_i = -1$ moments M_i and T_i with opposite signs.

From symmetry, we accordingly have:

$$\delta_{ik'} = 0$$

If, therefore, we substitute in equations 1 to 16 -

$$\delta_{a'a} = \delta_{b'a} = \delta_{c'a} = \ldots. = 0$$

$$\delta_{aa'} = \delta_{ba'} = \delta_{ca'} = \ldots. = 0$$

$$\delta_{a'b} = \delta_{b'b} = \delta_{c'b} = \ldots. = 0$$

etc.

we obtain two groups of eight equations, each with eight unknown quantities, of which the first group contains the unknown quantities X_a, X_b, X_c, etc., and the second group only the unknown quantities X'_a, X'_b, X'_c, etc. The first group agrees with the equations given at the commencement of the present paper while the second group is as follows:

1. $X'_a \quad \delta_{a'a'} + X'_b \quad \delta_{b'a'} + X'_c \quad \delta_{c'a'} + \ldots. = \Sigma P_m \quad \delta_{ma'}$,

2. $X'_a \quad \delta_{a'b'} + X'_b \quad \delta_{b'b'} + X'_c \quad \delta_{c'b'} + \ldots. = \Sigma P_m \quad \delta_{mb'}$,

3. $X'_a \quad \delta_{a'c'} + X'_b \quad \delta_{b'c'} + X'_c \quad \delta_{c'c'} + \ldots. = \Sigma P_m \quad \delta_{mc'}$.

etc. to 8.

As the loads P only produce bending moments in the spars and the loads $X'_a = -1$, $X'_b = -1$, $X'_c = -1$, etc. only produce bending moments in the ribs and torsional moments in the spars, we get

$$\Sigma P_m \; \delta_{ma'} = \Sigma P_m \; \delta_{mb'} = \Sigma P_m \; \delta_{mc'} \ldots = 0$$

The right hand sides of the last eight equations are, therefore, zero.

For the coefficients of the unknowns, we obtain, in accordance with the previous work:

$$E \; I \; \delta_{a'a'} = 2 \; \frac{4}{3} \int_0^s 1^2 \; dx + 4 \times 6 \int_0^s 1^2 \; dz = \frac{8}{3} s + 24 s = = \frac{80}{3} s,$$

$$E \; I \; \delta_{a'b'} = E \; I \; \delta_{a'c'} = E \; I \; \delta_{a'd'} = \ldots =$$

$$= 2 \; \frac{4}{3} \int_c^s 1^2 \, dx + 2 \times 6 \int_c^s 1^2 \, dz = \frac{44}{3} s,$$

$$E \; I \; \delta_{b'b'} = 2 \; \frac{4}{3} \int_0^{2s} 1^2 \, dx + 4 \times 6 \int_0^s 1^2 \, dz = \frac{16}{3} s +$$

$$+ 24 s = \frac{88}{3} s,$$

$$E \; I \; \delta_{b'c'} = E \; I \; \delta_{b'd'} = E \; I \; \delta_{b'e'} = \ldots =$$

$$= 2 \; \frac{4}{3} \int_c^{2s} 1^2 \, dx + 2 \times 6 \int_0^s 1^2 \, dz = \frac{52}{3} s,$$

etc.

Multiplying by 3/4 we obtain:

1. $20 X'_a + 11 X'_b + 11 X'_c + 11 X'_d + 11 X'_e + 11 X'_f +$

$$+ 11 X'_g + 11 X'_h = 0,$$

2. $$11\,X'_a + 22\,X'_b + 13\,X'_c + 13\,X'_d + 13\,X'_e + 13\,X'_f + {} + 13\,X'_g + 13\,X'_h = 0,$$

3. $$11\,X'_a + 13\,X'_b + 24\,X'_c + 15\,X'_d + 15\,X'_e + 15\,X'_f + {} + 15\,X'_g + 15\,X'_h = 0,$$

4. $$11\,X'_a + 13\,X'_b + 15\,X'_c + 26\,X'_d + 17\,X'_e + 17\,X'_f + {} + 17\,X'_g + 17\,X'_h = 0,$$

5. $$11\,X'_a + 13\,X'_b + 15\,X'_c + 17\,X'_d + 28\,X'_e + 19\,X'_f + {} + 19\,X'_g + 19\,X'_h = 0,$$

6. $$11\,X'_a + 13\,X'_b + 15\,X'_c + 17\,X'_d + 19\,X'_e + 30\,X'_f + {} + 21\,X'_g + 21\,X'_h = 0.$$

7. $$11\,X'_a + 13\,X'_b + 15\,X'_c + 17\,X'_d + 19\,X'_e + 21\,X'_f + {} + 32\,X'_g + 23\,X'_h = 0,$$

8. $$11\,X'_a + 13\,X'_b + 15\,X'_c + 17\,X'_d + 19\,X'_e + 21\,X'_f + {} + 23\,X'_g + 34\,X'_h = 0.$$

These equations can be reduced to the form:

1. $20X'_a - 9X'_b = 0,$

2. $-9X'_a + 20X'_b - 9X'_c = 0,$

3. $-9X'_b + 20X'_c - 9X'_d = 0,$

4. $-9X'_c + 20X'_d - 9X'_e = 0,$

. . . .

. . . .

. . . .

8. $-9X'_g + 20X'_h = 0.$

If we substitute in these equations:

$$
\begin{aligned}
X'_a &= A\ z_1 + B\ z_2, \\
X'_b &= A\ z_1^2 + B\ z_2^2, \\
X'_c &= A\ z_1^3 + B\ z_2^3, \\
&\cdots\cdots\cdots \\
&\cdots\cdots\cdots \\
&\cdots\cdots\cdots \\
X'_g &= A\ z_1^7 + B\ z_2^7, \\
X'_h &= A\ z_1^8 + B\ z_2^8.
\end{aligned}
$$

where z_1 and z_2 are the two real roots of the equation -

$$-9\ z^2 + 20\ z - 9 = 0$$

we obtain from equation 1,

$$20\ (A\ z_1 + B\ z_2) - 9\ (A\ z_1^2 + B\ z_2^2) = 0$$

or,

$$A\ (20\ z_1 - 9\ z_1^2) + B\ (20\ z_2 - 9\ z_2^2) = 0.$$

But, by assumption, we have -

$$20\ z_1 - 9\ z_1^2 = 20\ z_2 - 9\ z_2^2 = +\ 9$$

whence, A must equal $-B$.

We find from equation 8:

$$-9\ (A\ z_1^7 + B\ z_2^7) + 20\ (A\ z_1^8 + B\ z_2^8) = 0$$

or, with A equal to $-B$,

$$A\ [-9\ z_1^7 + 20\ z_1^8 + 9\ z_2^7 - 20\ z_2^8] = 0,$$

$$A\ [z_1^7\ (-9 + 20\ z_1) - z_2^7\ (-9 + 20\ z_2)] = 0.$$

But, by assumption:

$$-9 + 20\,z_1 = 9\,z_1^2,$$

$$-9 + 20\,z_2 = 9\,z_2^2,$$

therefore,

$$A\,(z_1^9 - z_2^9) = 0.$$

Since $(z_1^9 - z_2^9) = 0$ cannot become zero in this equation, we find that A and, consequently, also B must be zero. From this it follows that -

$$X'_a = X'_b = X'_c \,....\, X'_h = 0.$$

Translated by
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Table I.

	Bending of the spars.1.	Torsion of the spars.2.	Bending of the ribs. 3.	Results 1	Results 2	Results 3	Total
δ_{aa}	$2\cdot\int_0^s\left(\frac{7}{8}x\right)^2\cdot dx+2\cdot\int_0^{7s}\left(\frac{1}{8}x'\right)^2\cdot dx'$	$2\cdot\frac{4}{3}\int_0^s s^2\cdot dx$	$4\cdot 6\cdot\int_0^s z^2\cdot dz$	$\frac{49}{12}s^3$	$\frac{8}{3}s^3$	$8s^3$	$\frac{177}{12}s^3$
δ_{ab}	$2\cdot\int_0^s\frac{7}{8}\cdot\frac{6}{8}x^2\cdot dx+2\cdot\int_0^{6s}\frac{1}{8}\cdot\frac{2}{8}\cdot x'^2\cdot dx'+2\int_0^s\left(\frac{7}{8}s-\frac{1}{8}\xi\right)\cdot\frac{6}{8}\cdot(s+\xi)\cdot d\xi$	,,	$2\cdot 6\cdot\int_0^s z^2\cdot dz$	$\frac{81}{12}s^3$	,,	$4s^3$	$\frac{161}{12}s^3$
δ_{ac}	$2\cdot\int_0^s\frac{7}{8}\cdot\frac{5}{8}x^2\cdot dx+2\cdot\int_0^{5s}\frac{1}{8}\cdot\frac{3}{8}\cdot x'^2\cdot dx'+2\cdot\int_0^{2s}\left(\frac{7}{8}s-\frac{1}{8}\xi\right)\cdot\frac{5}{8}\cdot(s+\xi)\cdot d\xi$	,,	,,	$\frac{95}{12}s^3$	,,	,,	$\frac{175}{12}s^3$
δ_{ad}	$2\cdot\int_0^s\frac{7}{8}\cdot\frac{4}{8}\cdot x^2\cdot dx+2\cdot\int_0^{4s}\frac{1}{8}\cdot\frac{4}{8}\cdot x'^2\cdot dx'+2\cdot\int_0^{3s}\left(\frac{7}{8}s-\frac{1}{8}\xi\right)\cdot\frac{4}{8}\cdot(s+\xi)\cdot d\xi$	,,	,,	$\frac{94}{12}s^3$	,,	,,	$\frac{174}{12}s^3$
δ_{ae}	$2\cdot\int_0^s\frac{7}{8}\cdot\frac{3}{8}x^2\cdot dx+2\cdot\int_0^{3s}\frac{1}{8}\cdot\frac{5}{8}\cdot x'^2\cdot dx'+2\cdot\int_0^{4s}\left(\frac{7}{8}-\frac{1}{8}\xi\right)\cdot\frac{3}{8}\cdot(s+\xi)\cdot d\xi$	,,	,,	$\frac{81}{12}s^3$	,,	,,	$\frac{161}{12}s^3$
δ_{af}	$2\cdot\int_0^s\frac{7}{8}\cdot\frac{2}{8}x^2\cdot dx+2\cdot\int_0^{2s}\frac{1}{8}\cdot\frac{6}{8}x'^2\cdot dx'+2\cdot\int_0^{5s}\left(\frac{7}{8}-\frac{1}{8}\xi\right)\cdot\frac{2}{8}\cdot(s+\xi)\cdot d\xi$	,,	,,	$\frac{59}{12}s^3$	,,	,,	$\frac{139}{12}s^3$
δ_{ag}	$2\cdot\int_0^s\frac{7}{8}\cdot\frac{1}{8}x^2\cdot dx+2\cdot\int_0^{2s}\frac{1}{8}\cdot\frac{7}{8}x'^2\cdot dx'+2\cdot\int_0^{6s}\left(\frac{7}{8}-\frac{1}{8}\xi\right)\cdot\frac{1}{8}\cdot(s+\xi)\cdot d\xi$	,,	,,	$\frac{31}{12}s^3$	,,	,,	$\frac{111}{12}s^3$
δ_{ah}	0	,,	,,	0	,,	,,	$\frac{80}{12}s^3$
δ_{bb}	$2\cdot\int_0^{2s}\left(\frac{6}{8}x\right)^2\cdot dx+2\cdot\int_0^{6s}\left(\frac{2}{8}x'\right)^2\cdot dx'$	$2\cdot\frac{4}{3}\int_0^{2s}s^2\cdot dx$	$4\cdot 6\cdot\int_0^s z^2\cdot dz$	$\frac{144}{12}s^3$	$\frac{16}{3}s^3$	$8s^3$	$\frac{304}{12}s^3$
δ_{bc}	$2\cdot\int_0^{2s}\frac{6}{8}\cdot\frac{5}{8}\cdot x^2\cdot dx+2\cdot\int\frac{2}{8}\cdot\frac{3}{8}\cdot x'^2\cdot dx'+2\cdot\int_0^s\left(\frac{12}{8}s-\frac{2}{8}\xi\right)\cdot\frac{5}{8}\cdot(2s+\xi)\cdot d\xi$	,,	$2\cdot 6\cdot\int_0^s z^2\cdot dz$	$\frac{175}{12}s^3$	,,	$4s^3$	$\frac{287}{12}s^3$
δ_{bd}	$2\cdot\int_0^{2s}\frac{6}{8}\cdot\frac{4}{8}\cdot x^2\cdot dx+2\cdot\int\frac{2}{8}\cdot\frac{4}{8}x'^2\cdot dx'+2\cdot\int_0^{2s}\left(\frac{12}{8}s-\frac{2}{8}\xi\right)\cdot\frac{4}{8}\cdot(2s+\xi)\cdot d\xi$	,,	,,	$\frac{176}{12}s^3$	,,	,,	$\frac{288}{12}s^3$
δ_{be}	$2\cdot\int_0^{2s}\frac{6}{8}\cdot\frac{3}{8}x^2\cdot dx+2\cdot\int\frac{2}{8}\cdot\frac{5}{8}x'^2\cdot dx'+2\cdot\int_0^{3s}\left(\frac{12}{8}s-\frac{2}{8}\xi\right)\cdot\frac{3}{8}\cdot(2s+\xi)\cdot d\xi$	,,	,,	$\frac{153}{12}s^3$	,,	,,	$\frac{265}{12}s^3$
δ_{bf}	$2\cdot\int_0^{2s}\frac{6}{8}\cdot\frac{2}{8}x^2\cdot dx+2\cdot\int\frac{2}{8}\cdot\frac{6}{8}x'^2\cdot dx'+2\cdot\int_0^{4s}\left(\frac{12}{8}s-\frac{2}{8}\xi\right)\cdot\frac{2}{8}\cdot(2s+\xi)\cdot d\xi$	,,	,,	$\frac{112}{12}s^3$	,,	,,	$\frac{224}{12}s^3$
δ_{bg}	$=\delta_{1af}$	,,	,,	$\frac{59}{12}s^3$	,,	,,	$\frac{171}{12}s^3$
δ_{bh}	0	,,	,,	0	,,	,,	$\frac{112}{12}s^3$
δ_{cc}	$2\cdot\int_0^{3s}\left(\frac{5}{8}x\right)^2\cdot dx+2\cdot\int_0^{5s}\left(\frac{3}{8}x'\right)^2\cdot dx'$	$2\cdot\frac{4}{3}\int_0^{3s}s^2\cdot dx$	$4\cdot 6\cdot\int_0^s z^2\cdot dz$	$\frac{225}{12}s^3$	$\frac{24}{3}s^3$	$8s^3$	$\frac{417}{12}s^3$
δ_{cd}	$2\cdot\int_0^{3s}\frac{5}{8}\cdot\frac{4}{8}\cdot x^2\cdot dx+2\cdot\int_0^{4s}\frac{3}{8}\cdot\frac{4}{8}\cdot x'^2\cdot dx'+2\cdot\int_0^s\left(\frac{15}{8}s-\frac{3}{8}\xi\right)\cdot\frac{4}{8}\cdot(3s+\xi)\cdot d\xi$	,,	$2\cdot 6\cdot\int_0^s z^2\cdot dz$	$\frac{234}{12}s^3$	,,	$4s^3$	$\frac{378}{12}s^3$
δ_{ce}	$2\cdot\int_0^{3s}\frac{5}{8}\cdot\frac{3}{8}\cdot x^2\cdot dx+2\cdot\int_0^{3s}\frac{3}{8}\cdot\frac{5}{8}\cdot x'^2\cdot dx'+2\cdot\int_0^{2s}\left(\frac{15}{8}s-\frac{3}{8}\xi\right)\cdot\frac{3}{8}\cdot(3s+\xi)\cdot d\xi$	,,	,,	$\frac{207}{12}s^3$	,,	,,	$\frac{351}{12}s^3$
δ_{cf}	$=\delta_{1be}$	,,	,,	$\frac{153}{12}s^3$	,,	,,	$\frac{297}{12}s^3$
δ_{cg}	$=\delta_{1ae}$	,,	,,	$\frac{81}{12}s^3$	,,	,,	$\frac{225}{12}s^3$
δ_{ch}	0	,,	,,	0	,,	,,	$\frac{144}{12}s^3$

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Table I continued

	Bending of the spars.1.	Torsion of the spars.2.	Bending of the ribs.3.	Results 1	Results 2	Results 3	Total
δ_{dd}	$2 \cdot 2 \cdot \int_{0}^{4s} \left(\frac{4}{8} x\right)^2 \cdot dx$	$2 \cdot \frac{4}{3} \int_{0}^{4s} s^2 \cdot dx$	$4 \cdot 6 \cdot \int_{0}^{s} z^2 \cdot dz$	$\frac{256}{12} s^3$	$\frac{32}{3} s^3$	$8 s^3$	$\frac{480}{12} s^3$
δ_{de}	$= \delta_{1cd}$	,,	$2 \cdot 6 \cdot \int_{0}^{s} z^2 \cdot dz$	$\frac{234}{12} s^3$	,,	$4 s^3$	$\frac{410}{12} s^3$
δ_{df}	$= \delta_{1bd}$	,,	,,	$\frac{176}{12} s^3$	,,	,,	$\frac{352}{12} s^3$
δ_{dg}	$= \delta_{1ad}$	,,	,,	$\frac{94}{12} s^3$	,,	,,	$\frac{270}{12} s^3$
δ_{dh}	0	,,	,,	0	,,	,,	$\frac{176}{12} s^3$
δ_{ee}	$= \delta_{1cc}$	$2 \cdot \frac{4}{3} \int_{0}^{5s} s^2 \cdot dx$	$4 \cdot 6 \cdot \int_{0}^{s} z^2 \cdot dz$	$\frac{225}{12} s^3$	$\frac{40}{3} s^3$	$8 s^3$	$\frac{481}{12} s^3$
δ_{ef}	$= \delta_{1bc}$	,,	$2 \cdot 6 \cdot \int_{0}^{s} z^2 \cdot dz$	$\frac{175}{12} s^3$	,,	$4 s^3$	$\frac{383}{12} s^3$
δ_{eg}	$= \delta_{1ac}$	,,	,,	$\frac{95}{12} s^3$	,,	,,	$\frac{303}{12} s^3$
δ_{eh}	0	,,	,,	0	,,	,,	$\frac{208}{12} s^3$
δ_{ff}	$= \delta_{1bb}$	$2 \cdot \frac{4}{3} \int_{0}^{6s} s^2 \cdot dx$	$4 \cdot 6 \cdot \int_{0}^{s} z^2 \cdot dz$	$\frac{144}{12} s^3$	$\frac{48}{3} s^3$	$8 s^3$	$\frac{432}{12} s^3$
δ_{fg}	$= \delta_{1ba}$	,,	$2 \cdot 6 \cdot \int_{0}^{s} z^2 \cdot dz$	$\frac{81}{12} s^3$	,,	$4 s^3$	$\frac{321}{12} s^3$
δ_{fg}	0	,,	,,	0	,,	,,	$\frac{240}{12} s^3$
δ_{gg}	$= \delta_{1aa}$	$2 \cdot \frac{4}{3} \int_{0}^{7s} s^2 \cdot dx$	$4 \cdot 6 \cdot \int_{0}^{s} z^2 \cdot dz$	$\frac{49}{12} s^3$	$\frac{56}{3} s^3$	$8 s^3$	$\frac{369}{12} s^3$
δ_{gh}	0	,,	$2 \cdot 6 \cdot \int_{0}^{s} z^2 \cdot dz$	0	,,	$4 s^3$	$\frac{272}{12} s^3$
δ_{hh}	0	$2 \cdot \frac{4}{3} \int_{0}^{8s} s^2 \cdot dx$	$4 \cdot 6 \cdot \int_{0}^{s} z^2 \cdot dz$	0	$\frac{64}{3} s^3$	$8 s^3$	$\frac{352}{12} s^3$

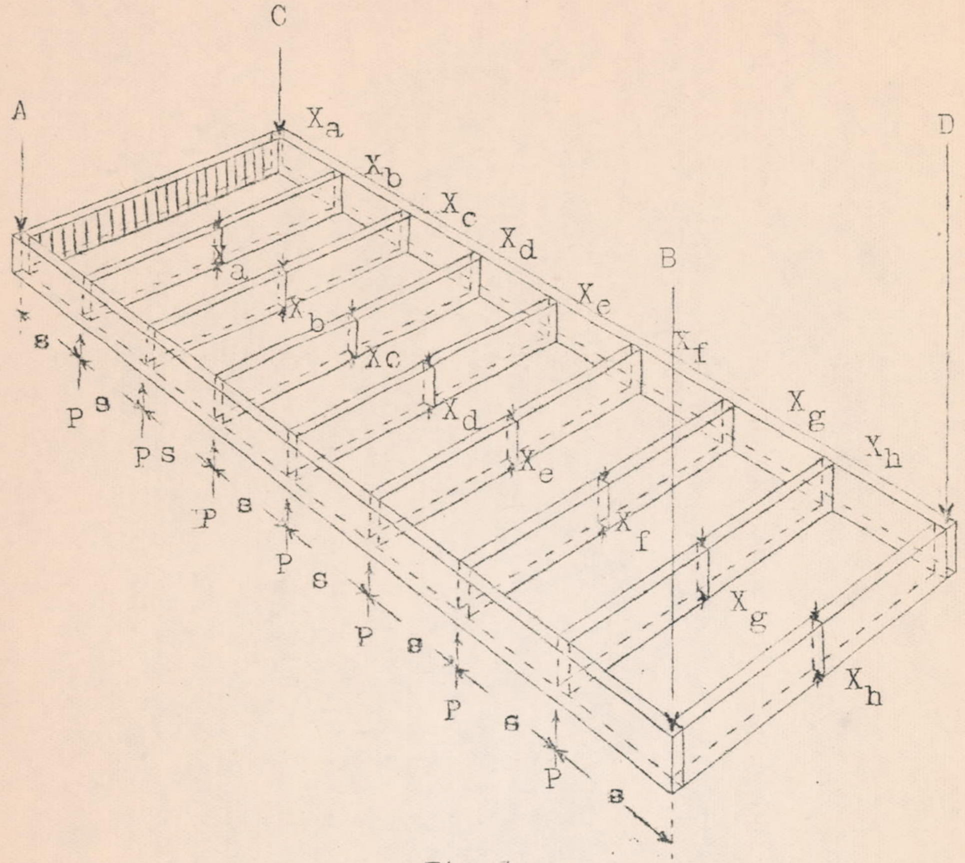

Fig.1.

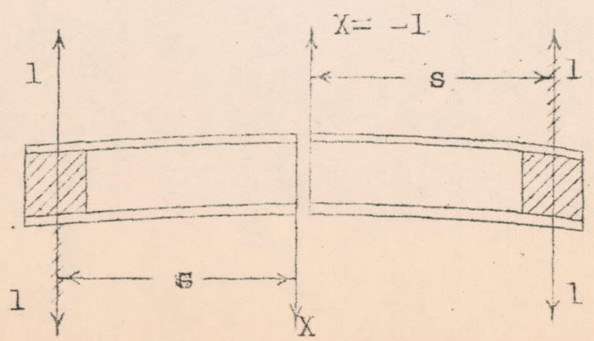

Fig.2.

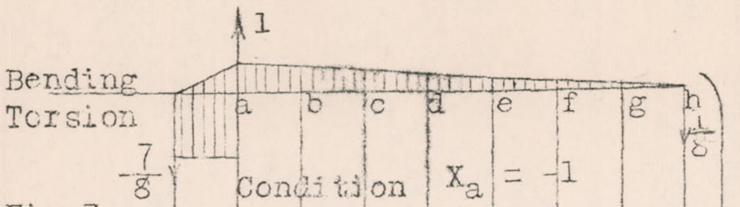

Fig.3.

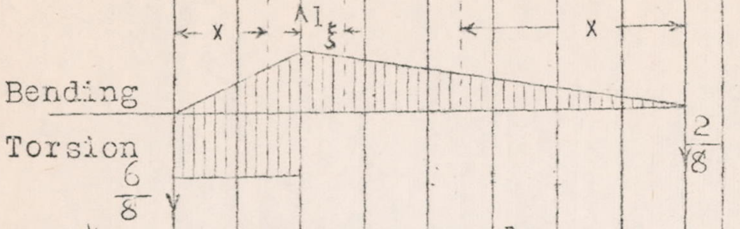

Fig.4.

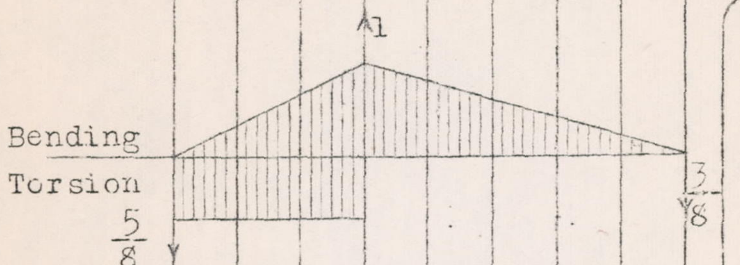

Fig.5.

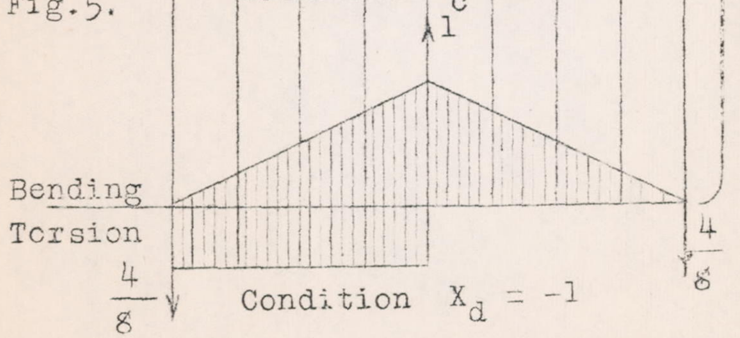

Fig.6.

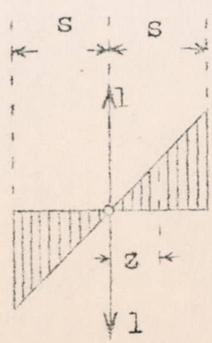

Fig.7.

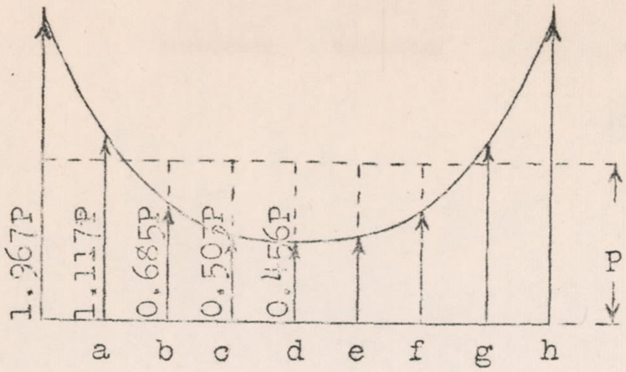

Fig.8.

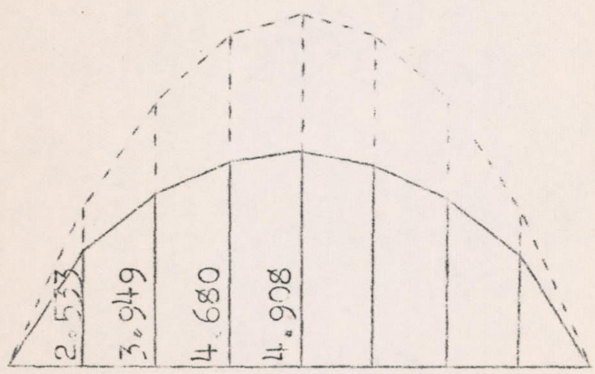

Fig.9.

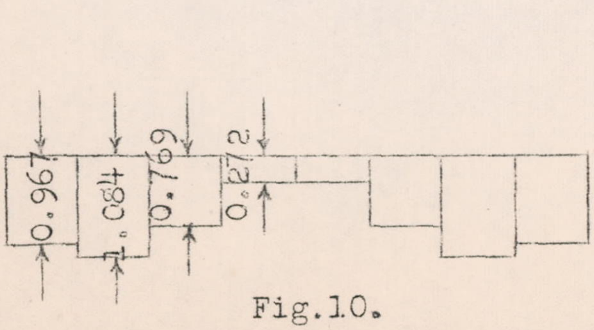

Fig.10.

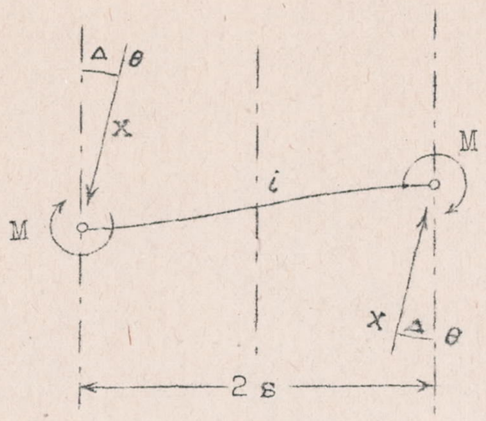

Fig.11.

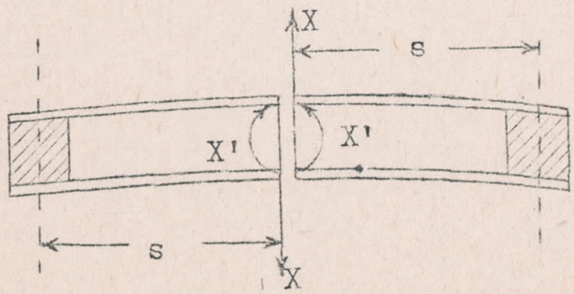

Fig.12.